Comments and Replies

Comments on "2-D Order-16 Integer Transforms for HD Video Coding"

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***Abstract*—In a recent paper, Dong *et al*. proposed a set of order-16 nonorthogonal integer cosine transforms (NICTs). They proved that the reconstruction error caused by the nonorthogonality is negligible as compared to the error caused by the quantization. However, we would like to point out three problems found in derivations and also give two comments. Nevertheless, the problems are defects only, hence do not affect the overall justifications to the proposed NICT. This letter is to enhance and clarify the proof of Dong *et al*.'s work.**

***Index Terms*—Error analysis, integer transform, nonorthogonal error, transform coding.**

I. Brief Review

In SECTION III-B of [1], the derivation starts by assuming an input signal as a 1-D, zero-mean, unit-variance first-order Markov process. It is denoted as an $N \times 1$ vector $\mathbf{x}$, and the correlation between neighboring elements is ρ. The perfect reconstruction is no longer preserved with the nonorthogonal kernel $\mathbf{T}$, even without the quantization. After the forward and inverse transforms $\boldsymbol{\theta} = \mathbf{Tx}$ and $\mathbf{y} = \mathbf{T^T}\boldsymbol{\theta}$, the reconstruction error can be modeled as $\mathbf{y} - \mathbf{x} = \mathbf{E_r}\ \mathbf{x}$, where $\boldsymbol{\theta}$ and $\mathbf{y}$ are the transformed and reconstructed vectors, respectively, T is the transposition, $\mathbf{E_r} = \mathbf{T^T T} - \mathbf{I}$, and $\mathbf{I}$ is the identity matrix. If the quantization is considered, the quantized-and-dequantized $\boldsymbol{\theta}$ is denoted as $\mathbf{u}$, the reconstructed vector is denoted as $\mathbf{y}_\mathrm{r}$, and the quantization error $\boldsymbol{\theta} - u$ is denoted as $\mathbf{q}$. Through a set of enlightening derivations, [1] obtained the following relationship:

$$\sigma_\mathrm{r}^2 = \sigma_\mathrm{q}^2 + \sigma_\mathrm{r0}^2 \tag{1}$$

where σ_r^2 and $\sigma_\mathrm{r0}{}^2$ are average variances of the reconstruction error with and without the consideration of quantization, respectively, and σ_q^2 is the quantization error. Reference [1] also obtained an upper bound for σ_r0^2 shown as follows:

$$\sigma_\mathrm{r0}^2 \le \frac{1}{N}\sigma_x^2 \sum_{k,j=0}^{N-1} M(k, j) \tag{2}$$

where σ_x^2 is the variance of the input signal and $\mathbf{M} = \mathbf{E_r}^\mathrm{T}\mathbf{E_r}$. Equation (1) reveals that the reconstruction error is simply the addition of the quantization and nonorthogonal error. The upper bounds of the proposed kernels were evaluated, and [1] concludes that the error introduced by the nonorthogonal transforms is negligible.

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II. Corrections

Let us point out three problems of [1]. First, $\mathbf{TT}^\mathrm{T}$ should not be replaced with $\mathbf{E}_\mathrm{r} + \mathbf{I}$ in (14) of [1] since $\mathbf{TT}^\mathrm{T}$ does not equal $\mathbf{T^T T}$. Instead, we define

$$\mathbf{E}_\mathrm{r}' = \mathbf{TT}^\mathrm{T} - \mathbf{I} \tag{3}$$

hence the results in [1, (14)] can be revised to

$$\sigma_\mathrm{r}^2 = \sigma_\mathrm{q}^2 + \sigma_\mathrm{r0}^2 + \frac{1}{N}\mathrm{E}\left[\mathbf{q}^\mathrm{T}\mathbf{E}_\mathrm{r}'\ \mathbf{q}\right] - \frac{2}{N}\mathrm{E}\left[\mathbf{q}^\mathrm{T}\mathbf{E}_\mathrm{r}'\ \boldsymbol{\theta}\right]. \tag{4}$$

This correction does not affect the final result of derivations as shown in (1), but it does affect the intermediate steps as shown below.

Second, the problem is about the derivations related to $\mathrm{E}[\mathbf{q}^\mathrm{T}\mathbf{E}_\mathrm{r}'\mathbf{q}] = 0$ and $\mathrm{E}[\mathbf{q}^\mathrm{T}\mathbf{E}_\mathrm{r}'\boldsymbol{\theta}] = 0$, as indicated by [1, (14) and (15)]. Let us quote a few words from [1], "because the autocorrelation of q and the crosscorrelation of q and θ are both zero as shown by Widrow *et al.*" recall that in Widrow *et al.*'s work [2], they only worked out the 1-D case and proved that the crosscorrelation between the quantizer input θ and its quantization noise q (a uniform independent noise) equals 0 provided that the quantizer input is "band-limited." This result is important, yet it only exploits the crosscorrelation in 1-D case. The idea that "the autocorrelation of q equals zero" stated by the authors is not fully correct since: 1) in 1-D case (a random variable q) the autocorrelation equals σ_q^2 at the origin and 0 elsewhere, or 2) in N-D case (a vector $\mathbf{q}$ composed of N random variables) the diagonal elements of autocorrelation matrix are the variances of random variables and the off-diagonal elements are all 0. Reference [1] dealt with the N-D case, hence the result from Widrow *et al.* is not sufficient. To begin with, we convert the third and fourth terms of (4) into (5) and (6), respectively, because the original form is not an algebraically convenient form for analysis

$$\begin{aligned}\mathrm{E}\left[\mathbf{q}^\mathrm{T}\mathbf{E}_\mathrm{r}'\ \mathbf{q}\right] &= \sum_{m,n=0}^{N-1} E_\mathrm{r}'(m,n)\,\mathrm{E}\left[q(m)\,q(n)\right] \\ &= \sum_{m,n=0}^{N-1} E_\mathrm{r}'(m,n)\,R_{qq}(m,n)\end{aligned} \tag{5}$$

$$\mathrm{E}\left[\mathbf{q}^{\mathrm{T}}\mathbf{E}_{\mathrm{r}}{}'\,\boldsymbol{\theta}\right] = \sum_{m,n=0}^{N-1} E_{\mathrm{r}}{}'(m,n)\,\mathrm{E}\left[q(m)\,\theta(n)\right] = \sum_{m,n=0}^{N-1} E_{\mathrm{r}}{}'(m,n)\,R_{q\theta}(m,n). \quad (6)$$

Note that $R_{qq}(m, n)$ is the (m, n)th element of the autocorrelation matrix of vector $\mathbf{q}$ which is composed of N random variables. The nth diagonal element is the variance of the nth random variable, whereas elements are zero elsewhere since each of them is the crosscorrelation of two different random variables [for example, $q(1)$ and $q(4)$ are two uniform independent noise processes]. Hence, (5) can be reduced to

$\mathrm{E}[\mathbf{q}^{\mathrm{T}}\mathbf{E}_{\mathrm{r}}'\mathbf{q}] = \sigma_{\mathrm{q}}^2 \Sigma_i E_{\mathrm{r}}'(i,i)$. By a careful examination of (3), it is observed that all diagonal terms [$E_{\mathrm{r}}{}'(i, i)$ for all i] are strictly zero, since $\mathbf{T}$ is a normalized matrix. Hence (5) is reduced to $\mathrm{E}[\mathbf{q}^{\mathrm{T}}\mathbf{E}_{\mathrm{r}}'\mathbf{q}] = 0$. Similarly, $R_{q\theta}(m, n)$ is the (m, n)th element of the matrix of the crosscorrelation between $\mathbf{q}$ and $\boldsymbol{\theta}$. The nth diagonal element is the crosscorrelation between quantization noise $q(n)$ and relative quantizer input $\theta(n)$. As we have mentioned before, by Widrow *et al.*'s work, $\mathrm{E}[q(n)\theta(n)] = 0$ in 1-D case. Hence we can approximate the diagonal elements to zero. The off-diagonal elements can also be approximated to zero. Each of them measures the crosscorrelation between the quantization noise of one quantizer $q(m)$ and the input of another quantizer $\theta(n)$. It is widely accepted that the DCT coefficients can be treated as uncorrelated random variables. Now that the inputs of two quantizers are not correlated, the input of one quantizer is also uncorrelated with the quantization noise of another quantizer. Hence the off-diagonal entries are also zero, and (6) can be further reduced to $\mathrm{E}[\mathbf{q}^{\mathrm{T}}\mathbf{E}_{\mathrm{r}}'\boldsymbol{\theta}] = 0$. We can thus conclude that the average variance of the reconstruction (4) can be reduced to $\sigma_{\mathrm{r}}^2 = \sigma_{\mathrm{q}}^2 + \sigma_{\mathrm{r0}}^2$, as stated in (1).

Third, although the upper bound of σ_{r0}^2 measured in terms of the order-8 kernel $\mathbf{T}_{8\mathrm{o}}$ (i.e., $\mathbf{T}_{8\mathrm{o}}$ is the submatrix of order-16 kernel $\mathbf{T}_{16}$) in Table III of [1] is useful for kernel comparison and selection, yet the use of this measurement as the justification of the negligibility of the nonorthogonal error $\sigma_{\mathrm{r0}}{}^2$ is just loosely defined. In order to make a concrete justification of the negligibility of σ_{r0}^2, we propose adding one more set of measurements which directly measure the upper bounds of σ_{r0}^2 for the order-16 kernels themselves ($\mathbf{T}_{16}$s). This set of measurements can then be compared to the quantization error σ_{q}^2, and subsequently the negligibility of σ_{r0}^2 can be confirmed in the strict sense. By using the same settings as in [1], i.e., $\sigma_x^2 = 1$, we have obtained the results for $\mathbf{T}_{16}$ as shown in column "Wc_measure" of Table I. It is observed that the errors measured for the order-16 kernels ($\mathbf{T}_{16}$s) are much smaller than that for their submatrices ($\mathbf{T}_{8\mathrm{o}}$s). It is reasonable since the other submatrix ($\mathbf{T}_{8\mathrm{e}}$) is orthogonal compared to the nonorthogonal submatrix ($\mathbf{T}_{8\mathrm{o}}$), hence the error can be diluted significantly.

III. Comments

Besides, let us give two more comments. First, the assumption of the input signal as a unit-variance Markov process is inappropriate. It is worth recalling that residual signals of the H.264/AVC encoder are being modeled, and these real data can hardly have a unit-variance. Actually, even if the variance of the input signal is not unit, the representation of the upper bound in (2) derived in the original work is still valid. We carried out a set of experiments to find the variances of input signals. It is found that the variances are of the order 10^2 to 10^3. By substituting these variances into (2) and making the use of column "Wc_measure" in Table I, we found that the errors caused by the nonorthogonal transform are in the order of 10^{-21} to 10^{-20} which are still much smaller than that caused by the quantization. This conclusion can be made safely by using the real variance of input signals.

TABLE I

UPPER BOUND OF AVERAGE VARIANCE OF RECONSTRUCTION ERROR WITHOUT THE CONSIDERATION OF QUANTIZATION

NICT Matrix Elements								Upper Bound of: 1) σ_{r0}^2 by setting $\sigma_x^2 = 1$, or 2) $\sigma_{\mathrm{r0}}^2/\sigma_x^2$		
								For $\mathbf{T}_{8\mathrm{o}}$	For $\mathbf{T}_{16}$	
x_1	x_3	x_5	x_7	x_9	x_{11}	x_{13}	x_{15}	Dong *et al.*	Wc_measure[1]	Proposed_wc
28	27	23	21	17	14	8	2	1.0849×10^{-6}	0 (-2.7967×10^{-23})	2.3868×10^{-6}
29	28	26	22	20	13	10	2	8.3628×10^{-7}	0 (-3.1019×10^{-23})	9.1990×10^{-7}
38	36	35	29	25	18	9	3	7.6103×10^{-7}	0 (-1.1410×10^{-23})	3.3368×10^{-6}
39	37	35	29	26	18	11	2	7.0311×10^{-7}	2.0913×10^{-23}	1.4603×10^{-6}
40	39	33	31	24	19	14	4	1.8527×10^{-6}	8.3438×10^{-23}	2.6789×10^{-6}
40	38	35	31	24	19	11	4	6.5425×10^{-7}	1.9439×10^{-23}	1.0820×10^{-6}

[1] The upper bounds of variance are set to zero for any negative value generated by the oversimplified and imprecise model as shown in (2).

Second, let us propose a better method compared to the method as shown in (2) to find the upper bound. In (2), one contributing factor of the upper bound is obtained by summing up all $M(k, j)$s which may be a positive or negative number. Hence, the cancelation effect occurs and one may wonder whether the assumption can reach the real worse-case scenario. What is the result if $R(k, j)$ approximates infinitely close to σ_x^2 and has the same sign with $M(k, j)$? This leads to the largest upper bound. To exploit this possibility, we do not make the assumption of wide-sense stationary (i.e., $M(k, j) \neq M(k-j, 0)$), which can give a flexible selection of the values of $M(k, j)$, and make the analysis using the Markov model.

It is well known that a zero-mean AR (1)-process can be denoted as $x(n) = \rho(n)x(n-1) + \varepsilon(n)(|\rho(n)| < 1)$ where $x(n)$ and $x(n-1)$ are adjacent random variables, $\rho(n)$ is the correlation coefficient between $x(n)$ and $x(n-1)$ and $\varepsilon(n)$ is a white noise process with zero mean. By removing the assumption of wide-sense stationary, the correlation between two variables $x(i)$ and $x(j)$ $(i < j)$ can be calculated as

$$\begin{aligned} R(i,j) &= \mathrm{E}\left[x(i)\,x(j)\right] \\ &= \mathrm{E}\left\{x(i)\cdot\left[\rho(j)\left[\rho(j-1)\left[\cdots\left[\rho(i+1)\,x(i)+\varepsilon(i+1)\right]\cdots\right] \right.\right.\right. \\ &\quad \left.\left.\left. +\varepsilon(j-1)\right]+\varepsilon(j)\right]\right\} \\ &= \left[\rho(j)\,\rho(j-1)\cdots\rho(i+1)\right]\sigma_i^2 \end{aligned} \quad (8)$$

where σ_i^2 is the variance of $x(i)$. It is seen that $R(i, j)$ is determined by a set of correlation coefficients. Let us denote $\rho(j)\cdot\rho(j-1)\cdot\ldots\cdot\rho(i+1)$ by $\rho(i, j)$. The worse-case scenario can be reached when: 1) $|\rho(i, j)| \to 1$, and 2) $sign\,[\rho(i, j)] =$

$$\mathbf{R} = \sigma_x^2 \begin{bmatrix} 1 & \rho(1) & \rho(2)\rho(1) & \rho(3)\rho(2)\rho(1) \cdots \rho(N-1)\cdots\rho(3)\rho(2)\rho(1) \\ \rho(1) & 1 & \rho(2) & \rho(3)\rho(2) \quad \cdots \rho(N-1)\cdots\rho(3)\rho(2) \\ \rho(2)\rho(1) & \rho(2) & 1 & \rho(3) \quad \cdots \rho(N-1)\cdots\rho(3) \\ \rho(3)\rho(2)\rho(1) & \rho(3)\rho(2) & \rho(3) & 1 \quad \ddots \vdots \\ \vdots & \vdots & \vdots & \ddots \quad \ddots \vdots \\ \rho(N-1)\cdots\rho(3)\rho(2)\rho(1) & \rho(N-1)\cdots\rho(3)\rho(2) & \rho(N-1)\cdots\rho(3) & \cdots \quad \cdots 1 \end{bmatrix} \quad (7)$$

sign $[M(i, j)]$. The first condition is easily fulfilled by setting the absolute value of all correlation coefficients to 1. The second condition can only be fulfilled for some positions. It is because the number of sign patterns of $M(i, j)$ is $2^{(N-1)^2/2}$, whereas that of $\rho(i, j)$ is only 2^{N-1}. Hence a computer search for all sign patterns of $\rho(i, j)$ can be carried out to find the best one which leads to the worst-case scenario. When the variance of noise approximates zero, we can use σ_x^2 instead of σ_i^2. The proof is shown as follows:

$$\begin{aligned} \mathrm{E}\left\{[x(n)]^2\right\} &= \mathrm{E}\left\{[\rho(n)\,x(n-1) + \varepsilon(n)]^2\right\} \\ \Rightarrow \sigma_n^2 &= [\rho(n)]^2\,\sigma_{n-1}^2 + \sigma_\varepsilon^2 \approx \sigma_{n-1}^2 \\ \Rightarrow \sigma_n^2 &= \sigma_{n-1}^2 = \cdots = \sigma_1^2 = \sigma_x^2. \end{aligned} \quad (9)$$

Hence the autocovariance matrix can be written as shown in (7) at the top of the page.

Actually, the worst-case scenario is reached when any two of random variable $x(n)$ have exactly the same observation ($\rho = 1$) or neighboring observations have exactly the opposite signs ($\rho = -1$). This can be infinitively approximated, but cannot be reached. Through computer searches for various kernels, we obtained new results as shown in the column "Proposed_wc" of Table I in this letter. The best sign patterns of $\rho(i, j)$ considered are not shown here for the sake of simplicity. Although the number of computer search is small compared to the possible number of sign patterns, it still needs 2^{15} different sign patterns for a length-16 input signal.

References

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